

Generalization of Fechner's Approach

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Abstract

Generalization of Fechner's assumptions, Weber's Law and constancy of subjective JNDs, are proposed to be included in deriving sensory scales, and the role of variance of sensation intensity is emphasized. Today, Weber's Law is accepted as an approximate one and its generalizations, the generalized Weber's law or the near-miss to Weber's law, are well known. Constancy of subjective JNDs seems to hold without questions, but this assumption presupposes constancy of variance of sensation intensity. Ekman's law denies equality of subjective JNDs. Differential equation, which is introduced in this paper, includes these generalizations of the two basic Fechnerian assumptions to relate them to sensory scale.

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Generalization of Fechner's Approach

フェヒナーの方法の一般化について

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Need of Generalization

Since Fechner proposed his famous theory of sensation, psychophysics, about one and a half centuries have passed. Current state of knowledge suggests generalization of his approach. It is explained that his theory is based on two assumptions, Weber's Law and constancy of subjective JNDs (e.g., Luce, 1993, p.133; Krueger, 1989, p.253). Since Weber's Law is concerned to JND, JND is basic concept in Fechner's theory. But actual size of JND is dependent on the discrimination probability adopted. Pointing out this arbitrariness of discrimination probability, Luce and Galanter (1963) proposed the revised Fechner problem. This formulation permits the use of differential equation (e.g., Okamoto (1993, 2001, 2003, e.g.); Dzhafarov and Coloniusⁱ (1999)). Okamoto's model can incorporate generalizations of Fechner's approach.

Generalization of Weber's law

Weber's Law, one of the two basic assumptions, is considered at the present time as the first approximation. Current popular generalizations of Weber's Law are the followings,

(a) The generalized Weber's Law

$$\Delta I = aI + b$$

and

(b) The near-miss to Weber's Law

$$\frac{\Delta I}{I} = aI^{-\mu},$$

where I and ΔI represent physical intensity of a stimulus and JND in terms of the physical intensity.

Luce (1993) says that a great deal of intensity data are well approximated by the generalized Weber's Law (ibid., p. 123) and that in case of pure tone discrimination, the near-miss to Weber's Law do well (ibid., p. 124). Luce (1993, p. 133) explains that the generalized Weber's Law with the assumption of equality of subjective JNDs leads to Fechner's Law,

$$\psi(I) = A \cdot \log(I + b/a) + B,$$

where $\psi(I)$ represents intensity of sensation of physical stimulus I .

Constancy of subjective JNDs

As to equality of subjective JNDs, i.e. sensation differences for physical JNDs, Thurstonian model shows importance of the role of variance of sensation in this equality assumption as follows:

Let the probability of discrimination between two physical intensities I_1 and I_2 be represented by $\Pr(I_1 < I_2)$, where $I_1 < I_2$ means that I_1 is judged to be weaker than I_2 . According to the Thurstonian framework, a simplified representation of this probability is as follows (Thurstone's Case III, 1927),

$$\Pr(I_1 < I_2) = \int_{-\infty}^{(\psi(I_2) - \psi(I_1)) / \sqrt{\sigma^2(I_1) + \sigma^2(I_2)}} \phi_0(x) dx, \quad (1)$$

where $\psi(I_1)$ and $\psi(I_2)$ represent the mean intensities of sensations evoked by the physical intensities I_1 and I_2 , $\sigma^2(I_1)$ and $\sigma^2(I_2)$ the variances of sensations by I_1 and I_2 , $\phi_0(x)$ the standard normal density distribution. Eq. (1) shows that when

$$\Pr(I_1 < I_2) = \text{const}, \quad \text{for } I_2 - I_1 = JND,$$

then

$$\frac{\psi(I_2) - \psi(I_1)}{\sqrt{\sigma^2(I_1) + \sigma^2(I_2)}} = \text{const}. \quad (2)$$

But, eq. (2) meansⁱⁱ that

$$\psi(I_2) - \psi(I_1) = \text{const}, \quad \text{i.e. differences of sensations are equal,}$$

$$\text{only in case of } \sqrt{\sigma^2(I_1) + \sigma^2(I_2)} = \text{const}.$$

Hence, when variance $\sigma^2(I)$ varies as I varies, subjective JND $\psi(I_2) - \psi(I_1)$ is not constant as I_1 varies.

Variability of variance $\sigma^2(I)$ of sensation has been proposed by several authors. For prothetic continua (e.g., loudness), Stevens (1975, p.235) proposes the principle of linear growth of subjective variability, which he calls Ekman's law,

$$\Delta\psi = k\psi \quad (\text{Ekman, 1959, p.345}),$$

where $\Delta\psi$ represents difference of sensation corresponding to JNDⁱⁱⁱ. Krueger (1989) discusses variability of subjective JND, i.e. $\Delta\psi$. Wickens (2002) gives statistical explanation of variability of variance of sensation.

By Ekman's law and eq. (2), we have

$$\text{const} = \frac{\psi(I_2) - \psi(I_1)}{\sqrt{\sigma^2(I_1) + \sigma^2(I_2)}} = \frac{\Delta\psi}{\sqrt{\sigma^2(I_1) + \sigma^2(I_2)}} \approx \frac{k}{\sqrt{2}} \cdot \frac{\psi}{\sqrt{\sigma^2(I_1)}}$$

Hence, we get

$$\sigma^2(I_1) \approx \alpha \cdot \psi^2,$$

where α is some constant.

That is, variance $\sigma^2(I_1)$ of sensation increases with intensity ψ of sensation. It is shown in the next section that variability of variance of sensation plays a critical role in construction of a subjective scale. Neglecting the role of variance of sensation would invite troubles. For example, Dzhaferov and Colonius' (1999) model does not include this factor of variance, and leads to trouble (See appendix A).

A Variable Variance Model

Generalization of Fechner's assumptions, Weber's Law and constancy of sensation differences for JNDs, can be incorporated in one equation, which is explained in this section. This psychophysical equation is derived by combining two types of intensities, physical and sensory ones, by discrimination probability (Okamoto, 1993, 2001, 2003). Relation of physical intensity and discrimination probability can be represented by a psychometric function. Discrimination probability can be also described as a function of sensory intensities by Thurstonian modeling.

A possible form of psychometric function can be put as follows

$$\Pr(I_2 \succ I_1) = \int_{-\infty}^{\frac{1}{k(I_1)} \frac{I_2 - I_1}{I_1} + o(I_2 - I_1)} \phi_0(x) \cdot dx \quad (3)$$

In the above equation, $\phi_0(x)$ represents the standard normal distribution function. The assumption that discrimination probability can be described by the cumulative normal distribution function will be discussed with detailed explanation of eq. (3), after the following Thurstonian model is presented

$$\Pr(I_2 \succ I_1) = \int_{-\infty}^{(\psi(I_2) - \psi(I_1)) / \sqrt{\sigma^2(I_2) + \sigma^2(I_1) - 2r\sigma(I_1)\sigma(I_2)} + o(\psi(I_2) - \psi(I_1))} \phi_0(x) \cdot dx, \quad (4)$$

where $\psi(I_1)$ and $\psi(I_2)$ are means of sensory intensities evoked by physical intensities I_1 and I_2 . Sensory intensities are assumed to have variances $\sigma^2(I_1)$ and $\sigma^2(I_2)$ with correlation coefficient r between them (Thurstone, 1927).

The use of the normal distribution in the above equations can be justified as a working frame, or we can consider normal distributions as the first approximation. Justification as a working frame is based on the technical convenience that employment of normal distributions simplifies mathematical manipulations. Simplification by employing a special frame is also observed in physics. Newtonian physics is built on inertial frames (Arya, 1979). Normal distributions play the same role in psychophysics as inertial frames in Newtonian physics.

Normal distributions as the first approximation also work well in the above equations. The usage of normal distribution as an approximation entails inclusion of little oh notation $o(x)$ (Apostol, 1961) in the equations. The little oh notation in the above equations represents any function which satisfies the following condition:

$$\frac{o(x)}{x} \rightarrow 0 \quad \text{as } x \rightarrow 0$$

Inclusion of the little oh notation into eqs. (3) and (4) means that representation by normal distributions holds at least asymptotically around the point of subjective equality (PSE). It should be noted that, when Weber's Law holds strictly, the term $o(I_2 - I_1)$ is needed in the psychometric function eq. (3) (Okamoto, 2001; Drosler, 2000).

We can relax the normality assumption and express the equations in more general form, as is shown in appendix B.

Combining eqs. (3) and (4), we have

$$\begin{aligned} \Pr(I_2 \succ I_1) &= \int_{-\infty}^{\frac{1}{k(I_1)} \cdot \frac{I_2 - I_1}{I_1} + o(I_2 - I_1)} \phi_0(x) \cdot dx \\ &= \int_{-\infty}^{(\psi(I_2) - \psi(I_1)) / \sqrt{\sigma(I_2)^2 + \sigma(I_1)^2 - 2r\sigma(I_1)\sigma(I_2)} + o(\psi(I_2) - \psi(I_1))} \phi_0(x) \cdot dx \end{aligned}$$

Hence, we have

$$\frac{1}{k(I_1)} \cdot \frac{I_2 - I_1}{I_1} + o(I_2 - I_1) = \frac{(\psi(I_2) - \psi(I_1))}{\sqrt{\sigma(I_2)^2 + \sigma(I_1)^2 - 2r\sigma(I_1)\sigma(I_2)}} + o(\psi(I_2) - \psi(I_1))$$

Multiplying both sides of the above equation by

$$\frac{\sqrt{\sigma(I_2)^2 + \sigma(I_1)^2 - 2r\sigma(I_1)\sigma(I_2)}}{I_2 - I_1},$$

we have

$$\begin{aligned} &\frac{\sqrt{\sigma(I_2)^2 + \sigma(I_1)^2 - 2r\sigma(I_1)\sigma(I_2)}}{k(I_1) \cdot I_1} \\ &+ \frac{o(I_2 - I_1)}{I_2 - I_1} \cdot \frac{\sqrt{\sigma(I_2)^2 + \sigma(I_1)^2 - 2r\sigma(I_1)\sigma(I_2)}}{I_2 - I_1} \\ &= \frac{(\psi(I_2) - \psi(I_1))}{I_2 - I_1} \\ &+ \frac{o(\psi(I_2) - \psi(I_1))}{(\psi(I_2) - \psi(I_1))} \cdot \frac{(\psi(I_2) - \psi(I_1))}{(I_2 - I_1)} \cdot \frac{\sqrt{\sigma(I_2)^2 + \sigma(I_1)^2 - 2r\sigma(I_1)\sigma(I_2)}}{I_2 - I_1} \end{aligned}$$

Let

$$I_2 \rightarrow I_1$$

with appropriate regularity conditions and replace I_1 by I , we have

$$\frac{\sqrt{2(1-r)} \cdot \sigma(I)}{k(I) \cdot I} = \frac{d\psi}{dI} \quad (5)$$

Eq. (5) includes explicitly two terms, $\sigma(I) = \sqrt{\sigma(I)^2}$ and $k(I)$. By the term $\sigma(I)$, we can relax the assumption of constant variance of sensation, so incorporate Ekman's Law into eq. (5), e.g. By the term $k(I)$, we can incorporate various generalizations of Weber's Law into eq. (5). Relation of $k(I)$ to Weber function is shown in the next section.

Weber Function and $k(I)$

Let $z_{0.75}$ and $I_{0.75}$ be the values corresponding to discrimination probability 0.75 such that

$$0.75 = \int_{-\infty}^{z_{0.75}} \phi_0(t) \cdot dt = \Pr(I_{0.75} > I) = \int_{-\infty}^{\frac{1}{k(I)} \frac{I_{0.75} - I}{I} + o(I_{0.75} - I)} \phi_0(t) \cdot dt$$

Then we have

$$z_{0.75} \approx \frac{1}{k(I)} \cdot \frac{I_{0.75} - I}{I}$$

$$k(I) \approx \frac{1}{z_{0.75}} \cdot \frac{I_{0.75} - I}{I}$$

Because $\frac{I_{0.75} - I}{I}$ is Weber fraction for discrimination probability 0.75 and $\frac{1}{z_{0.75}}$

constant coefficient, $k(I)$ can be considered to be essentially Weber function.

When we adopt the generalized Weber's Law, we put

$$k(I) \propto \frac{\Delta I}{I} = a + b \cdot I^{-1}$$

When we adopt the near-miss to Weber's Law, we have

$$k(I) \propto \frac{\Delta I}{I} = a \cdot I^{-\mu}$$

Some Special Cases

Selecting specific forms of functions $\sigma(I)$ and $k(I)$, we can derive the corresponding psychophysical function $\psi(I)$ by solving eq. (5). Some special cases are treated in this section. Firstly, Fechner's assumptions are applied to eq. (5). Then, current laws as to Weber functions, i.e. the generalized Weber's Law and the near-miss to Weber's Law, are applied. In these cases where Weber's Law is generalized, the assumption of constancy of subjective JNDs is also replaced by Ekman's law or something like this.

(1) Fechner's assumptions:

Weber's Law means that

$$k(I) = C = \text{const}$$

and equality of sensation differences for JNDs is represented as

$$\sigma(I) = D = \text{const}$$

Substituting $k(I)$ and $\sigma(I)$ in eq. (5) by the above two equations, we have

$$\frac{\sqrt{2(1-r)} \cdot D}{C \cdot I} = \frac{d\psi}{dI}$$

$$\frac{d\psi}{dI} = A \cdot \frac{1}{I},$$

where

$$A = \frac{\sqrt{2(1-r)} \cdot D}{C} = \text{const}$$

Solving the above differential equation, we have

$$\psi = A \log I + B,$$

where $B = \text{const}$.

Fechner's assumptions lead to Fechner's law (i.e. a logarithmic function).

(2) The generalized Weber's Law and Ekman's Law:

The generalized Weber's Law can be represented as

$$k(I) = a + b \cdot I^{-1}$$

and Ekman's Law as

$$\sigma(I) = c \cdot \psi(I). \quad (6)$$

Substituting $k(I)$ and $\sigma(I)$ in eq. (5) by the above two equations, we have

$$\frac{\sqrt{2(1-r)} \cdot c \psi}{(a + bI^{-1}) \cdot I} = \frac{d\psi}{dI}$$

$$\frac{d\psi}{dI} = \frac{\sqrt{2(1-r)} \cdot c \cdot \psi}{a(I + b/a)}$$

Solving the above equation, we have

$$\log \psi = \gamma \cdot \log(I + b/a) + B,$$

where $\gamma = \frac{\sqrt{2(1-r)} \cdot c}{a} = \text{const}$ and $B = \text{const}$

Hence, we have

$$\psi = \alpha(I + \beta)^\gamma$$

where $\alpha = e^B = \text{const}$, $\beta = b/a = \text{const}$.

The above equation is called Stevens' (power) law (Luce, 1993).

(3) The near-miss to Weber's Law and Ekman's Law:

The near-miss to Weber's Law can be represented as

$$k(I) = aI^{-\mu}. \quad (7)$$

Substituting $k(I)$ and $\sigma(I)$ in eq. (5) by eqs. (7) and (6), we have

$$\frac{\sqrt{2(1-r)} \cdot c \psi}{aI^{-\mu} \cdot I} = \frac{d\psi}{dI}$$

$$\frac{d\psi}{dI} = A \psi I^{\mu-1}$$

where $A = \frac{\sqrt{2(1-r)} \cdot c}{a} = \text{const}$.

Solving the above equation, we have

$$\log \psi = \frac{A}{\mu} I^\mu + B$$

where $B = \text{const}$.

Hence we have,

$$\psi = \exp\left(\frac{A}{\mu} I^\mu + B\right)$$

In the next case, Ekman's law is slightly modified into a little different from than eq. (6).

(4) Modified Ekman's Law:

In this case, Ekman's Law is modified and represented as

$$\sigma(I) = cI^d. \quad (8)$$

Substituting $k(I)$ and $\sigma(I)$ in eq. (5) by eqs. (7) and (8), we have

$$\frac{\sqrt{2(1-r)} \cdot cI^d}{aI^{-\mu} \cdot I} = \frac{d\psi}{dI}$$

$$\frac{d\psi}{dI} = \alpha \cdot I^\beta$$

where $\alpha = \frac{\sqrt{2(1-r)} \cdot c}{a} = \text{const}$ and $\beta = d + \mu - 1 = \text{const}$.

Solving the above equation, we have

$$\psi = \frac{\alpha}{\beta + 1} \cdot I^{\beta+1} + \gamma, \quad (9)$$

where $\gamma = \text{const}$.

Again, we get Stevens' power law. We may call eq. (8) near-miss to Ekman's law as we call eq. (7) the near-miss to Weber's Law. That is, the near-miss to Weber's Law and near-miss to Ekman's law lead to Stevens' power law.

Discussion

Update of Fechner's approach is considered. Fechner's approach is based on the two assumptions, Weber's Law and equality of subjective JNDs. Current generalizations of Weber's Law are the generalized Weber's Law and the near-miss to Weber's Law (Luce, 1993). The other assumption, equality of subjective JNDs, is given mathematical formulations such as the revised Fechner problem or the equally-often-noticed-difference problem (Luce and

Galanter, 1963). In this paper, it is pointed out that this assumption is dependent on constancy of variance of sensation and doubtful when we consider Ekman's law. But Ekman's law is derived, using the scale of sensation magnitude (Ekman, 1959, 1956). So Ekman's law is dependent on validity of the sensation scale. As is pointed out later in this section, some authors doubt the validity of the scales by direct methods.

Updating Fechner's approach leads to eq. (5). In derivation of eq. (5), normal distributions are employed. This usage of normal distributions can be justified as a frame in which mathematical operations become simple. In this sense, normal distribution is a component of eq. (5). Thurstone (1927, pp. 274 – 275) introduces into his models normal distribution as the one by which the scale is defined. But, in deriving eq. (5), normality assumption is required to hold asymptotically, and in appendix B it is shown that this assumption can be relaxed.

To add to the specification that derivation of eq. (5) is based on normality assumption, it should be pointed out that generalizations of Weber's Law hold on with respect to a specified physical unit. That is, these laws are meaningful of type-I (Niederee, 1994, p. 542; also see Okamoto, 1999; cf. Narens and Mausfeld, 1992). Hence, eq. (5) is also meaningful of type-I in the sense that it depends on the physical unit and the distribution. But these restrictions of meaningfulness are justified as the frame which makes mathematical operations easier like the inertial frame in Newtonian physics (Arya, 1979).

There are other forms of psychometric functions than normal distributions, e.g. logistic distributions, Weibull distributions and so on (Treutwein and Strasburger, 1999; Treutwein, 1995). But, Klein (2001, p. 1423) says that normal distributions play fundamental role in many approaches to psychometric functions. As to the dependence of eq. (5) on normal distributions, appendix B shows that we can derive essentially the same equation as eq. (5), even if other than normal distribution is assumed.

Eq. (5) is concerned with scales based on discrimination data. Stevens (1975, e.g.) proposes another type of scales, which are constructed from the data of direct estimation of magnitude or ratio of magnitudes of sensation. He criticizes Fechner's law. For example, Stevens (1936, p. 412) says "it has been found that summing the same number of DL's for two tones of different frequency does not yield equal loudness". But, this criticism about number of JNDs (DL's in the above citation) presupposes constancy of variance of sensation. If variances of sensation varies as stimulus intensity, then his criticism is not justified. Furthermore, empirical validity of direct method is questioned experimentally by Garner (1954). Laming (1997) proposes that psychological primitive in subject's psychophysical judgment is categorical. Baird (1997) shows that Stevens' power law can be generated in the simulation,

where the subject reports his estimation of intensity of the stimulus according to his preference to numbers (Number Preference Model). Before discussing relation between eq. (5) and direct methods, e.g. magnitude estimation scale, we must assure the validity of scales of direct methods.

References

- Apostol, T. M. (1961). *Calculus, Vol. 1*. Blaisdell Publishing Company, Inc.
- Arya, A. P. (1979). *Introductory college physics*. Machillan Publishing Co., Inc.
- Baird, J. C. (1997). *Sensation and judgment: Complementarity theory of psychophysics*. Mahwah: Lawrence Erlbaum Associates, Publishers.
- Drosler, J. (2000). An n-dimensional Weber law and the corresponding Fechner law. *Journal of Mathematical Psychology*, 44, 330 – 335.
- Dzhafarov, E. N. and Colonius, H. (1999). Fechnerian metrics in unidimensional and multidimensional stimulus spaces. *Psychonomic Bulletin & Review*, 6, 239 - 268.
- Ekman, G. (1956). Discriminal sensitivity on the subjective continuum. *Acta Psychologica*, 12, 233-243.
- Ekman, G. (1959). Weber's law and related functions. *The Journal of Psychology*, 47, 343 – 352.
- Falmagne, J.-C. (1985). *Elements of psychophysical theory*. Oxford University Press.
- Garner, W. R. (1954). Context effects and the validity of loudness scales. *Journal of Experimental Psychology*, 48, 218 – 224.
- Klein, S. A. (2001). Measuring, estimating, and understanding the psychometric function: A commentary. *Perception & Psychophysics*, 63, 1421 – 1455.
- Krueger, L. E. (1989). Reconciling Fechner and Stevens: Toward a unified psychophysical law. *Behavioral and Brain Sciences*, 12, 251 – 320.
- Laming, D. (1973). *Mathematical psychology*. Academic Press.
- Laming, D. (1997). *The measurement of sensation*. Oxford: Oxford University Press.
- Luce, R. D. (1993). *Sound & hearing: A conceptual introduction*. Hillsdale: Lawrence Erlbaum Associates, Publishers.
- Luce, R. D. and Galanter, E. (1963). Discrimination. In R. D. Luce, R. R. Bush and E. Galanter (Eds.), *Handbook of mathematical psychology: Vol. 1* (pp. 191 - 243). New York: John Wiley and Sons, Inc.
- Narens, L. and Mausfeld, R. (1992). On the relationship of the psychological and the physical

- in psychophysics. *Psychological Review*, 99, 467-479.
- Niederee, R. (1994). There is more measurement than just measurement: Measurement theory, symmetry, and substantive theorizing. *Journal of Mathematical Psychology*, 38, pp. 527 – 594.
- Okamoto, Y. (1993). *Just-noticeable difference and sensation scale*. Poster session presented at the annual meeting of the Japanese Psychonomic Society, Tokyo, Japan. (Summary of the presentation, *The Japanese Journal of Psychonomic Science*, 1993, 12, 60, in Japanese)
- Okamoto, Y. (1999). Some precautions in analyses of acoustic data. *The Journal of the Acoustical Society of Japan*, 55, 723-729. (in Japanese)
- Okamoto, Y. (2001). On Fechner's Problem. *Studies and Essays, Behavioral Sciences and Philosophy, Faculty of Letters, Kanazawa University*, No. 21, pp. 1 - 25.
- Okamoto, Y. (2003). A variable variance model: A critical formulation of Fechner's problem. In: H. Yanai, A. Okada, K. Shigemasa and J. J. Meulman (Eds.) *New Developments in Psychometrics: Proceedings of the international meeting of the psychometric society IMPS2001*. (pp. 535 - 542). Tokyo: Springer-Verlag.
- Stevens, S. S. (1936). A scale for the measurement of a psychological magnitude: Loudness. *Psychological Review*, 43, 405 – 416.
- Stevens, S. S. (1975). *Psychophysics: Introduction to its perceptual, neural, and social prospects*. New York: John Wiley & Sons.
- Treutwein, B. (1995). Adaptive psychophysical procedures. *Vision Research*, 35, 2503 – 2522.
- Treutwein, B. and Strasburger, H. (1999). Fitting the psychometric function. *Perception & Psychophysics*, 61, 87 – 106.
- Thurstone, L. L. (1927). A law of comparative judgment. *Psychological Review*, 34, 273-286.
- Wickens, T. D. (2002). *Elementary signal detection theory*. Oxford University Press.

Appendix A

A Problem of Dzhafarov and Colonius (1999) Model

Dzhafarov et al. (1999) use the following notation

$$\gamma_x(y) = \Pr(y \succ x)$$

where x and y denote physical intensities and $y \succ x$ means that y is judged to be more intense than x .

Putting

$$F(x) = \frac{d\gamma_x(y)}{dy} \Big|_{y=x},$$

Dzhafarov et al. (1999) define the Fechnerian distance $G(a, b)$ between stimulus a and b ($a < b$) as follows

$$G(a, b) = G(b, a) = \int_a^b F(x) dx.$$

Consider the following case, where Stevens' power law

$$\psi(x) = x^\alpha, \quad (\text{A.1})$$

and Ekman's law

$$\sigma(x) = \beta \cdot \psi = \beta \cdot x^\alpha,$$

hold.

Adopting Thurstone's (1927) Case III, we can put

$$\gamma_x(y) = \int_{-\infty}^{(\psi(y) - \psi(x)) / \sqrt{\sigma^2(y) + \sigma^2(x)}} \phi_0(t) \cdot dt,$$

where $\phi_0(t)$ is the standard normal distribution. Differentiating the above equation, we have

$$\begin{aligned} \frac{d\gamma_x(y)}{dy} &= \phi_0 \left(\frac{y^\alpha - x^\alpha}{\beta \cdot \sqrt{y^{2\alpha} + x^{2\alpha}}} \right) \cdot \frac{1}{\beta^2 (y^{2\alpha} + x^{2\alpha})} \\ &\quad \times \left\{ \alpha \beta \cdot y^{\alpha-1} \cdot \sqrt{y^{2\alpha} + x^{2\alpha}} - (y^\alpha - x^\alpha) \cdot \frac{\beta}{2} \cdot \frac{2\alpha \cdot y^{2\alpha-1}}{\sqrt{y^{2\alpha} + x^{2\alpha}}} \right\}. \end{aligned}$$

Hence, we have

$$\begin{aligned} F(x) &= \frac{d\gamma_x(y)}{dy} \Big|_{y=x} \\ &= \phi_0(0) \cdot \frac{\alpha}{\sqrt{2} \cdot \beta} \cdot x^{-1} \\ &= \delta \cdot x^{-1}, \end{aligned}$$

where

$$\delta = \frac{\phi_0(0) \cdot \alpha}{\sqrt{2} \cdot \beta}$$

So, we have

$$\begin{aligned} G(a, b) &= \int_a^b \delta \cdot x^{-1} \cdot dx \\ &= \delta \log b - \delta \log a. \end{aligned} \quad (\text{A.2})$$

Comparing eqs. (A.2) with (A.1), we see that the Fechnerian distance eq. (A.2) given by Dzhafarov et al. (1999) does not reproduce the sensory scale eq. (A.1), which is one of the most interesting types of scales in psychophysics.

Appendix B

More General Approach

In eqs. (3) and (4), the standard normal distribution $\phi_0(x)$ is employed. But, to derive eq. (5), normal distribution $\phi_0(x)$ is not needed. More generally, we can proceed as follows:

Put

$$\Pr(I_2 \succ I_1) = T \left\{ \frac{\psi(I_2) - \psi(I_1)}{[\sigma^2(I_1) + \sigma^2(I_2) - 2r\sigma(I_1)\sigma(I_2)]^{1/2}} \right\} \quad (\text{B.1})$$

and

$$\Pr(I_2 \succ I_1) = W \left(\frac{1}{k(I_1)} \cdot \frac{I_2 - I_1}{I_1} \right), \quad (\text{B.2})$$

where $T(z)$ and $W(z)$ are some cumulative distribution functions^{iv}, i.e.

$$\begin{aligned} \lim_{z \rightarrow -\infty} T(z) &= \lim_{z \rightarrow -\infty} W(z) = 0, \\ \lim_{z \rightarrow +\infty} T(z) &= \lim_{z \rightarrow +\infty} W(z) = 1, \end{aligned}$$

and it is assumed that $T(z)$ and $W(z)$ are not decreasing, and strictly increasing in the neighborhood of 0. $\sigma(I_1)$ and $\sigma(I_2)$ in eqs. (B.1) and (B.2) denotes some indices of dispersion, e.g. variances or any other appropriate ones, of sensations evoked by physical intensities I_1 and I_2 .

By eqs. (B.1) and (B.2), we have

$$\frac{\psi(I_2) - \psi(I_1)}{[\sigma^2(I_1) + \sigma^2(I_2) - 2r\sigma(I_1)\sigma(I_2)]^{1/2}} = T^{-1} \left\{ W \left[\frac{1}{k(I_1)} \cdot \frac{I_2 - I_1}{I_1} \right] \right\} \quad (\text{B.3})$$

with I_2 in the neighborhood of I_1 , where T^{-1} is defined.

When $h(x) = T^{-1}[W(x)]$ is asymptotically linear in the neighborhood of 0, i.e.,

$$h(x) = T^{-1}[W(x)] = \alpha \cdot x + o(x),$$

we have

$$\frac{\psi(I_2) - \psi(I_1)}{[\sigma^2(I_1) + \sigma^2(I_2) - 2r\sigma(I_1)\sigma(I_2)]^{1/2}} = \alpha \cdot \frac{1}{k(I_1)} \cdot \frac{I_2 - I_1}{I_1} + o(I_2 - I_1)$$

Let I_2 approach I_1 and replace I_1 by I , we have

$$\frac{d\psi}{dI} = \frac{\alpha \cdot \sigma(I) \cdot \sqrt{2(1-r)}}{k(I) \cdot I}$$

This equation is essentially the same as eq. (5). But, by using normal distribution in eq. (3), we can easily interpret $k(I)$ as Weber function.

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- i) In 1995, Colonius read one of earlier versions of my manuscript on the model discussed here. From 1994 to 1995, R. D. Luce also gave me stimulating comments on the earlier version of the model discussed here (see Okamoto, 2001, pp. 7-8).
 - ii) Falmagne (1985, pp. 135-136, 144-145) explains that in case of more general form of function σ can be linear function of ψ rather than constant. But, in this case, discrimination probability is represented as a function of difference of logarithms of σ s.
 - iii) $\Delta\psi$ denotes difference of mean sensations $\psi(I_2) - \psi(I_1)$, and ψ simply represents mean sensation $\psi(I_1)$ for stimulus I_1 .
 - iv) In this paper, a fundamental (Klein, 2001, p. 1422) or true psychometric function is considered.